

Hyperbolic Geometry and Its Metric in the Upper Half-Plane and Poincare Disk Models

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Abstract

Hyperbolic geometry is a non-Euclidean geometry defined by its violation of Euclid's parallel postulate, which states that for any point not on a given line, exactly one line can be drawn through that point parallel to the given line. In contrast, hyperbolic geometry permits infinitely many lines through a point that do not intersect a given line. This geometry is visualized on surfaces known as the hyperbolic plane, represented through models such as the Poincare Disk Model, the Upper Half-Plane Model, and the Klein Model, each offering a distinct representation of hyperbolic space. The hyperbolic metric, which defines distances between points, differs significantly from the Euclidean distance. Notably, triangles in hyperbolic geometry have sum of interior angles less than 180° , and the concept of parallel lines differ from Euclidean principles. These unique characteristics make hyperbolic geometry essential for understanding spaces with distinct properties. Distances in hyperbolic models are calculated using their respective metrics, and the analytical representation of hyperbolic lines has been established through examples in Poincare Disk Model, the Upper Half-Plane Model.

Keywords: Hyperbolic geometry, hyperbolic metric, Poincare disk model and upper half-plane model

1. Introduction

Since high school we have been familiar with geometry. In the high school geometry curriculum, we study triangles, circles, geometric constructions, proportions, etc. The geometry with which we are familiar is known as Euclidean Geometry, developed about 300 B.C. by Euclid in his famous books *Element*. In "Element", to prove his argument Euclid had included terminologies, definitions and postulates, and then by using these terms he proved propositions and theorems.

Euclid defined some terminologies as;

- A point is that which has no part.
- A line is breadthless length.
- The extremities of a line are points.
- A surface is that which has length and breadth.
- The extremities of a surface are lines.
- A plane angle is the inclination to one another of two lines in a plane and do not lie in a straight line.
- When the lines containing the angles are straight the angle is called rectilinear.
- A boundary is that which an extremity of anything is.

Euclid had given proper direction for the development of contemporary geometry [1]. For use, he introduced that some theorem must be accepted without justification, and termed them as postulates. Euclid's postulates are;

- 1) Every pair of distinct points can be joined by a straight line.
- 2) A straight line can be extended continuously in each of two directions.
- 3) One circle and only one can be drawn from any point as center and given line segment as radius.
- 4) All right angles are equal to each other.
- 5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right

angles. The two straight lines, if extended indefinitely, meet on that side on which the angles are less than two right angles.

In these days it is restate as:

If any point not in a given line, then one and only one parallel line can be drawn through given point to the given line.

Many Mathematicians accepted Euclid's first four postulates are self-evident and they tried to prove later postulate. Especially they tried to develop it as proposition. But they only succeed to develop following equivalent conditions;

- 1) Two parallel lines are equidistance (Posiclosius 1st, century B.C.).
- 2) If a line intersects one of two parallels, it also intersects the other (Proclus (410–485) A.D.).
- 3) Given a triangle, we can construct a similar triangle of any size (Wallis 1616–1703).

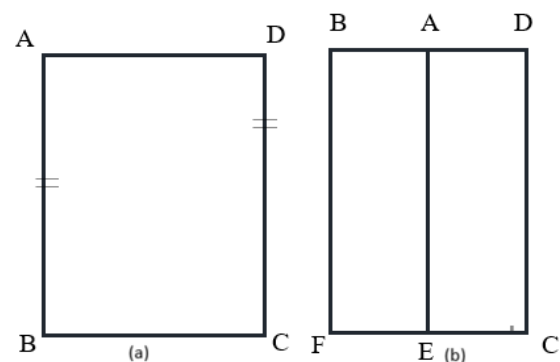


Figure 1. (a) Full and (b) Half Quadrilateral Concept

To prove Euclid's fifth postulate, Italian mathematician Saccheri (1661–1733) worked in quadrilateral $ABCD$, where $AB = DC$ and $\angle B = \angle C = 90^\circ$. He accepted the angles at

A and D (Figure 1) are equal and considered three hypotheses that they were classified as acute, obtuse or right.

In fact, he intended to show former two hypotheses lead to contradiction and later hypothesis able to prove the fifth postulate. He concluded that the hypothesis of obtuse angles implies fifth postulate. Based on the hypothesis of acute angles he deduced many results, which in these days has known as a part of hyperbolic geometry.

Fifty years later Lambert (1728–1777) also independently worked in the half of Saccheri's quadrilateral $ABFE$ [1]. He also rejected the hypothesis of obtuse angle and gave few ways for hypothesis of acute angle. He also introduced the idea of defect of polygon, which became main tool of hyperbolic geometry.

Meanwhile famous German Mathematician Gauss also denied the fifth postulate but he did not express his views until others published them [2]. Many mathematicians tried to find a proof by contradiction. They either assumed that there were at least two lines parallel or no lines parallel to the original line under the given conditions. These two tactics gave rise to two new geometries, hyperbolic (at least two lines parallel) and elliptic (no lines parallel). At the time of these discoveries, it was believed that these two geometries were false, and many people refused to believe that geometry other than Euclidean existed.

In 1829 A.D. Russian mathematician Lobachevski independently present the paper on the fallacy of fifth postulate and claimed that to a given line a pair of parallel lines can be drawn through the given point, which is not in given line [2]. This paper gave formal introduction to new geometry.

In this way the hyperbolic geometry arose as a separate discipline of geometry and different because of one of its very natural analytic models. Under this article two analytic models of hyperbolic space as given by Anderson [2]; Saul Stahl in Keen and Lakic [?] with some results given by Keen and Lakic [?], and Kobayashi [3] are included. Each model has its own metrics, lines called geodesics, isometrics and so on.

2. Theoretical Frameworks

To understand hyperbolic geometry, we need to know about a model and how distances are measured in it. In fact, a model is like a map that shows hyperbolic geometry. Two common models are the Poincare Disk and the Upper Half-Plane. These models are defined on parts of the Euclidean plane and give them a special metric, which is a way to measure distance.

A metric space is a set of points with a rule, called a metric, that tells us the distance between any two points. This rule must follow certain conditions:

- 1) Distances are never negative.
- 2) Only the same point has zero distance to itself.
- 3) The distance from A to B is the same as from B to A (Figure 1).
- 4) The direct distance between two points is always shorter than going through a third point. [3] [4] [5]

In hyperbolic geometry Riemannian metric can be used, which is a way to measure distances of every point on a curved surface. This helps to find the shortest path between two points, which we use as the distance.

The Poincare Disk and Upper Half-Plane models can be treated in hyperbolic geometry using tools from math, like complex numbers and calculus. Even though these models are built in a flat Euclidean space, their special metrics make them behave like hyperbolic spaces. The lines in these models are not straight lines like in Euclidean geometry, but they are curves in general and sometimes straight lines that represent the shortest paths in the hyperbolic world. [6] [7]

The Poincare Disk model is like a flat, round disk which is the region bounded by a circle, like a plate. The Upper Half-Plane model is like the top half of a flat plane which is above the x -axis in a plane. Both of these models are drawn in a flat Euclidean space, but their special metrics make them act like hyperbolic spaces. These metrics change the geometry inside these models so that they follow the rules of hyperbolic geometry instead of regular flat geometry [8].

3. Methodology

Non-Euclidean geometry was defined based on an alternative to the parallel postulate. So the fundamentals of hyperbolic geometry are based on the following postulate: Given a line l and a point P not on l , there exist at least two parallel lines to l through P (Figure 2).

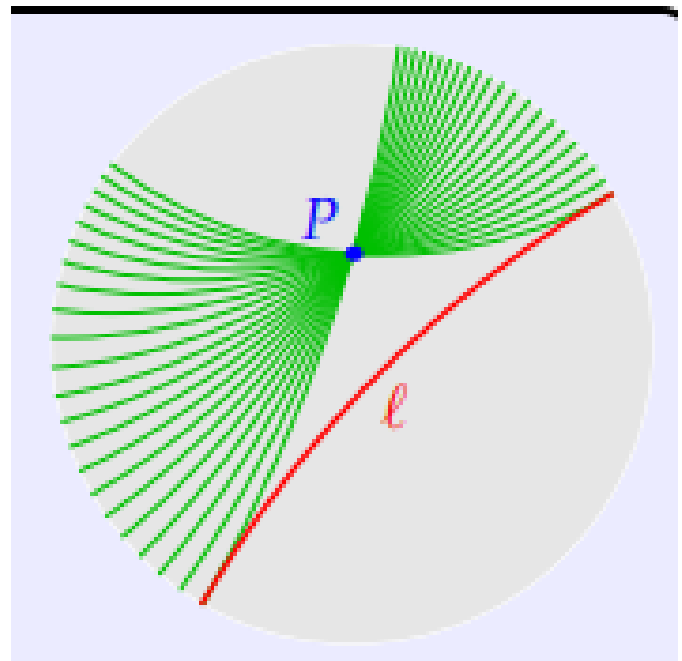


Figure 2. Concept of Hyper Geometric

3.1 Hyperbolic Metric

The hyperbolic metric refers to a way of measuring distances in hyperbolic geometry. The hyperbolic metric is typically defined in specific models of hyperbolic geometry, such as the Poincare disk model and the upper half-plane model. To

measure the length of a curve on a plane surface Riemann proposed in 1854 a general modification of distance [1]. He suggested new method to study geometries in which the length of an arbitrary curve γ lying on the plane would be given by the expression

$$\int_{\gamma} \sqrt{E dx^2 + 2F dx dy + G dy^2} \quad (1)$$

where E, F, G are the functions of x and y that satisfy conditions, $E > 0$, $G > 0$ and $EG - F^2 > 0$. The associated Riemann metrics that complete the analytic description of these two models are applicable to determine length of curve.

Remarks

1. If $E = G = 1$ and $F = 0$ in (1.1) then Riemann metric gives Euclidean length of curve γ .
2. If $E = G = \frac{1}{y^2}$ and $F = 0$, then metric gives hyperbolic length of curve γ .

3.2 Poincare Half-Plane Model

The Poincare half-plane model, $\mathbb{H}$ takes half of the Euclidean plane, as determined real line $\mathbb{R}$, to be its hyperbolic plane. (The line $\mathbb{R}$ is not actually included in this plane.) Lines are either semicircles orthogonal to $\mathbb{R}$ or rays perpendicular to $\mathbb{R}$. In the following figure, the hyperbolic lines in $\mathbb{H}$ are illustrated:

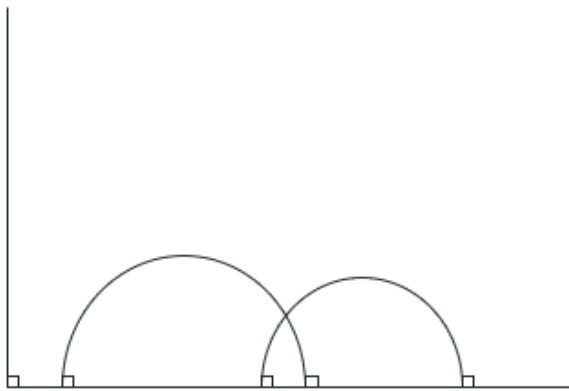


Figure 3. Geodesics in Hyperbolic Lines

So, there are two distinct types of hyperbolic lines and both are also defined in terms of Euclidean objects in $\mathbb{C}$. One is intersection of $\mathbb{H}$ with Euclidean line in $\mathbb{C}$ and perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$. The other is the intersection of $\mathbb{H}$ with a Euclidean circle centered on the real axis $\mathbb{R}$.

Note1: The geodesics in $\mathbb{H}$ /hyperbolic lines are the semi-circles orthogonal to the real axis and the vertical straight lines.

Note 2: As in the Euclidean Geometry for each pair of distinct points in $\mathbb{H}$, there exist unique hyperbolic line passing through them i.e. for given any two points in $\mathbb{H}$ there exists a unique geodesic passing through them [2].

Hyperbolic Length between two points Poincare half-plane model

The hyperbolic length of point $P(x, y)$ and $Q(x + dx, y + dy)$ is defined by $\frac{\sqrt{dx^2 + dy^2}}{y}$, where $\sqrt{dx^2 + dy^2}$ is the Euclidean length of points $P(x, y)$ and $Q(x + dx, y + dy)$.

In brief, in a plane; hyperbolic length = $\frac{\text{Euclidean length}}{y}$

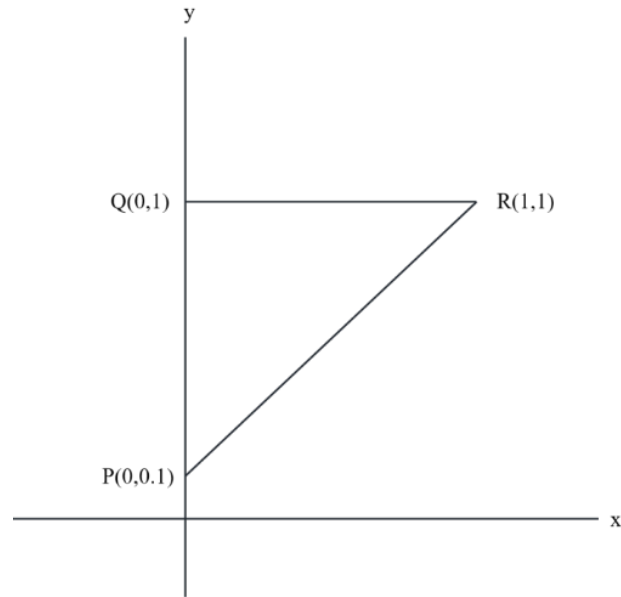


Figure 4. Geodesics in Hyperbolic Lines

For the points $P(0, 0.1)$, $Q(0, 1)$ and $R(1, 1)$ in a given plane, the hyperbolic length of PQ is;

$$\begin{aligned} &= \int_{0.1}^1 \frac{dy}{y} \\ &= \log_e y \Big|_{0.1}^1 \\ &= \log_e 1 - \log_e 0.1 \\ &= 2.303(\text{approx.}) \end{aligned} \quad (2)$$

$$\begin{aligned} \text{Hyperbolic length of } PR &= \int_0^1 \frac{\sqrt{dx^2 + (0.9dx)^2}}{0.9x + 0.1} \\ &= \int_0^1 \frac{\sqrt{1.81} dx}{0.9x + 0.1} \\ &= \frac{\sqrt{1.81}}{0.9} \int_0^1 \frac{0.9 dx}{0.9x + 0.1} \\ &= \frac{\sqrt{1.81}}{0.9} \log_e (0.9x + 0.1) \Big|_0^1 \\ &= \frac{\sqrt{1.81}}{0.9} \times 2.303 (\text{approx.}) \\ &= 3.442 (\text{approx.}) \end{aligned} \quad (3)$$

$$\begin{aligned}
 \text{Hyperbolic length of } QR &= \int_0^1 \frac{\sqrt{dx^2}}{1} \\
 &= \int_0^1 dx \\
 &= 1.
 \end{aligned} \quad (4)$$

Above in Figure, hyperbolic length of sides of triangle PQR are 1, 2.303 and 3.442 and $1 + 2.303 < 3.442$. So, it does not obey the triangular inequality, which is one fact which shows Euclidean and Hyperbolic Geometry differ [9].

And another interesting fact is that the sum of the interior angles of a hyperbolic triangle is less than two right angles.

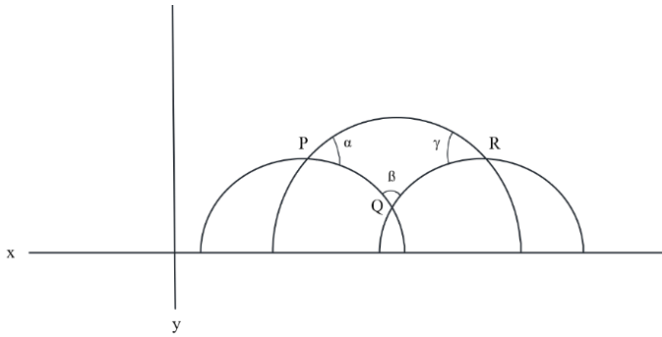


Figure 5. Hyperbolic Length

In the Figure 5, $\alpha + \beta + \gamma < \pi$ is called the deficiency and it is the area of hyperbolic triangle PQR. i.e. area of hyperbolic triangle $PQR = \pi - (\alpha + \beta + \gamma)$. Hence, if the deficiency of hyperbolic triangle is equal, then the area is also equal whatever space it occupies.

3.3 Hyperbolic length of arc in Poincare half-plane model

The hyperbolic length of arc PQ is the length of geodesics joining P and Q. For, given circle centered at $C(c, 0)$ and with radius 'r',

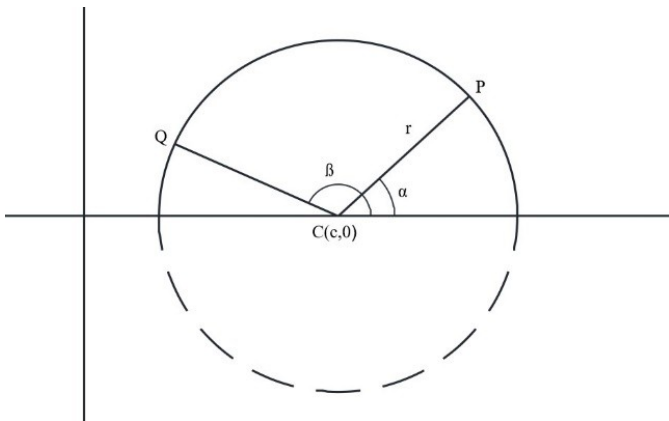


Figure 6. Hyperbolic Length of Arc

To find hyperbolic length of arc PQ. Here, CQ and CP makes angles β and α ($\alpha < \beta$), respectively with positive

x-axis. If θ be the angle made by radius through any point (x, y) then

$$\begin{aligned}
 x = c + r \cos \theta &\Rightarrow dx = -r \sin \theta d\theta, \\
 y = r \sin \theta &\Rightarrow dy = r \cos \theta d\theta.
 \end{aligned}$$

Hyperbolic length of PQ is;

$$\begin{aligned}
 &\int_{\alpha}^{\beta} \frac{\sqrt{(-r \sin \theta d\theta)^2 + (r \cos \theta d\theta)^2}}{r \sin \theta} \\
 &= \int_{\alpha}^{\beta} \frac{\sqrt{r^2 d\theta^2 (\sin^2 \theta + \cos^2 \theta)}}{r \sin \theta} \\
 &= \int_{\alpha}^{\beta} \frac{r d\theta}{r \sin \theta} \\
 &= \int_{\alpha}^{\beta} \csc \theta d\theta.
 \end{aligned} \quad (5)$$

On integration,

$$\text{Hyperbolic length of } PQ = \log_e \left(\frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha} \right) \quad (6)$$

3.4 The Poincare Disk Model

The Poincare disk model Δ (or conformal disk model) uses the interior of a circle as a plane. Lines are represented by arcs of circles that are orthogonal to the boundary circle, as well as diameters of the boundary circle. Hyperbolic lines in the Poincare disk model are illustrated in the adjacent figure.

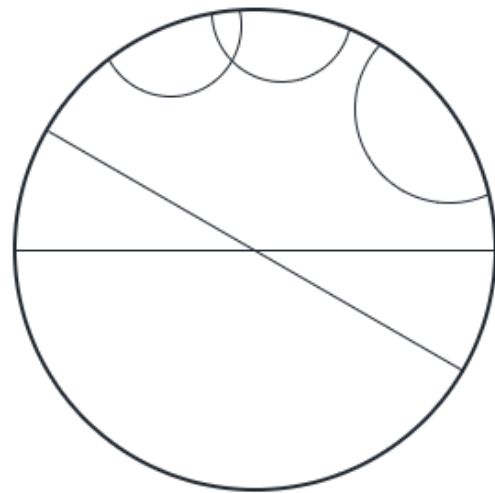


Figure 7. The Poincare Disk Model

Thus,

$$h = \{(x, y) : y > 0\}$$

and

$$\Delta = \{(x, y) : x^2 + y^2 < 1\}.$$

Note: Every hyperbolic line in the Poincare disk model is one of the following:

- 1) A diameter passing through $(c, d) \neq (0, 0)$.
- 2) The arc of a Euclidean circle with equation

$$x^2 + y^2 - 2ax - 2by + 1 = 0, \quad \text{where } a^2 + b^2 > 1.$$

and (Euclidean) center and radius

$$C = (a, b) \quad \text{and} \quad r = \sqrt{a^2 + b^2 - 1}.$$

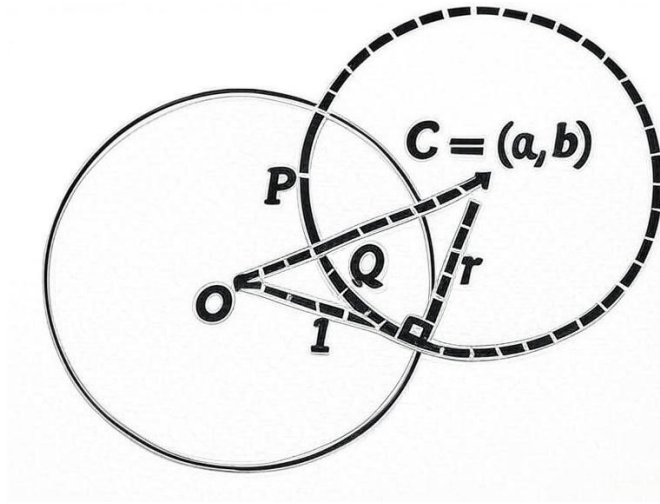


Figure 8. The Poincare Disk Model

Example: For the hyperbolic line through $P = (\frac{1}{3}, \frac{1}{2})$ and $Q = (\frac{1}{2}, 0)$ in the Poincare disk:

Substitute $P = (\frac{1}{3}, \frac{1}{2})$ and $Q = (\frac{1}{2}, 0)$ into

$$x^2 + y^2 - 2ax - 2by + 1 = 0,$$

we get

$$\frac{1}{9} + \frac{1}{4} - \frac{2}{3}a - b + 1 = 0$$

and

$$\frac{1}{4} - a + 1 = 0 \quad \Rightarrow \quad (a, b) = \left(\frac{5}{4}, \frac{19}{36}\right).$$

The hyperbolic line PQ has equation

$$x^2 + y^2 - \frac{5}{2}x - \frac{19}{18}y + 1 = 0,$$

or,

$$\left(x - \frac{5}{4}\right)^2 + \left(y - \frac{19}{36}\right)^2 = \frac{545}{648}.$$

The hyperbolic distance in the Poincare disk model

The hyperbolic distance between points P and Q in the Poincaré disk is

$$d(P, Q) = \cosh^{-1} \left(1 + \frac{2|PQ|^2}{(1 - |P|^2)(1 - |Q|^2)} \right) \quad (7)$$

where $|PQ|$ is the Euclidean distance and $|P|$, $|Q|$ are the Euclidean distances of P and Q from the origin.

Example: If we calculate the hyperbolic (geodesic) distance of an isosceles right triangle with vertices $O(0,0)$, $P(\frac{1}{2},0)$, and $Q(0,\frac{1}{2})$,

$$\begin{aligned} |P| &= \frac{1}{2} = |Q|, & |PQ|^2 &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \\ d(P, Q) &= \cosh^{-1} \left(1 + \frac{2 \cdot \frac{1}{2}}{(1 - \frac{1}{4})^2} \right) \\ &= \cosh^{-1} \left(\frac{25}{9} \right) \approx 1.681. \end{aligned}$$

4. Conclusion

The Poincare Disk and Upper Half-Plane models provide invaluable tools for performing calculations in hyperbolic geometry. Each model possesses computational advantages depending on the specific problem. The Poincare Disk Model represents the entire hyperbolic plane is contained within a finite Euclidean disk. This makes it easier to analyze the characteristics. But the hyperbolic lines sometimes be less straightforward to work with analytically compared to the vertical lines. The Upper Half-Plane Model, conversely, has the advantage that vertical lines are hyperbolic geodesics. Both models clearly illustrate the profound implications of the hyperbolic metric.

Hyperbolic geometry and its metric are serving in the space with constant negative curvature, contrasting with the zero curvature of Euclidean space. The hyperbolic metric in the upper half-plane model is associated with complex analysis and complex dynamics. The hyperbolic metric's ability to compute shortest paths through geodesics applicable network analysis in non-Euclidean environments or for optimizing paths in complex, curved spaces. The hyperbolic metric allows for more efficient representation and shortest path calculations, so it has applications in technology and data science. These applications are subjects of

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